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Application of the Leslie Matrix Model in Predicting Population Growth Rates and Livestock Harvesting

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Abstract

Leslie matrix model is a population growth model that can be used to predict the number and growth rate of populations that are female in population and female in animals. In animal populations, the Leslie matrix can be used in harvesting. This study aims to apply the Leslie matrix model to predict the number and growth rate of female cattle and determine the proportion of female cattle population harvesting. The female cattle populations used were female dairy cattle and female beef cattle. The results showed that the prediction of female dairy cattle population in 2022 - 2025 decreased every year. As for the female beef cattle population, the prediction results show that the number population always increase every year. Furthermore, to determine harvesting is only applied to the female beef cattle population. For uniform harvesting, the result is 11.4% of the population of each class and for the youngest class, the result is 50.5% of the population of the first age class.

Keywords: Leslie matrix, population growth, female cattle, harvesting

1. Introduction

Population growth is a crucial aspect in the study of population dynamics. It provides valuable information about future changes in population size—whether it will increase, decrease, or remain stable (Sibly & Hone, 2002). Demographers commonly employ the Leslie matrix model as a population growth model to predict future population size and growth rates (Jenkins, 1988). This model is specifically applied to represent the growth of female populations, as females are responsible for reproduction, and reproduction leads to an increase in the number of individuals within a population.

The Leslie matrix can be used to forecast the number of individuals in each age class in the subsequent period, provided that the current population distribution by age structure is known (Leslie, 1945). Additionally, it allows researchers to determine whether the population growth rate is increasing, decreasing, or remaining constant by calculating the dominant eigenvalue of the Leslie matrix (Cull & Vogt, 1973). If the growth rate of the female population in a species is increasing, or the dominant eigenvalue of the Leslie matrix is greater than one, the matrix can be further used to calculate harvesting rates (Mollet & Cailliet, 2002). Harvesting, in this context, refers to the removal or extraction of a portion of the population for other purposes (Anton & Rorres, 2014).

Livestock farming is one of the key sectors in Indonesia's agricultural industry. The livestock sub-sector plays an essential role in providing national food needs, such as meat and dairy products (Asresie et al., 2015). Among livestock commodities, cattle are the primary source of animal protein demanded by the public. This is due to the fact that cattle have the highest population, are widely distributed across regions, and consist of various breeds (Hall & Ruane, 1993). The government has implemented various programs to develop the livestock sector with the aim of increasing livestock production. To support these programs and meet the growing demand for cattle breeding stock, one viable strategy is to predict the population of cattle and apply appropriate harvesting policies. Such policies must be carefully designed to ensure the sustainability of the cattle population post-harvest and to avoid potential shortages. Based on the aforementioned considerations, this study aims to explore the application of the Leslie matrix in predicting the population size and growth rate of female cattle, as well as in determining the appropriate harvesting proportion of the female cattle population.

2. Materials and Methods

This study applies the Leslie matrix model to predict population growth and determine harvesting proportions. The data utilized include the female dairy cattle population at BBPTU-HPT Baturraden and the female beef cattle population at BPTU-HPT Sembawa from 2021 to 2022. The steps in this research comprise a literature review, data collection, determining fertility rates (a_i) and survival rates (b_i) of the population, constructing the Leslie matrix, developing the population growth model to forecast population growth rates, and determining harvesting proportions for the female cattle population.

2.1. Leslie Matrix

The Leslie matrix is a widely used model in demography for studying population growth. The model was introduced by the ecologist P. H. Leslie in 1945 and describes the population growth of humans and animals, focusing on the female population. The general form of the Leslie matrix is expressed as follows:

$$\mathbf{L} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{k-1} & 0 \end{bmatrix},$$

with a_i is defined as the average number of females born to individuals in age class i and b_i is the survival rate, which is the ratio of females surviving to age class $i+1$ at time $(t+1)$, to the number in age class i, at time t (Prayanti et al., 2021).

Let the female population at time *t* be divided into k age classes, where $n_1(t)$ represents the number of females in the first age class, $n_2(t)$ in the second, and so on, until $n_k(t)$ ain the *k*-th class. The total female population at time *t* is given by:

$$n(t) = n_1(t) + n_2(t) + \cdots + n_k(t). \tag{1}$$

The age distribution vector of the female population at time *t* is written as:

$$\mathbf{n}(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ \vdots \\ n_k(t) \end{bmatrix}.$$

This vector $\mathbf{n}(t)$ is referred to as the initial age distribution vector, which serves as the basis for predicting the female population in the next p periods, i.e., $n(t+p)$.

At time $t+1$, the total population is:

$$n(t+1) = n_1(t+1) + n_2(t+1) + \cdots + n_k(t+1). \tag{2}$$

The age distribution vector at time $t+1$ becomes:

$$\boldsymbol{n}(t+1) = \begin{bmatrix} n_1(t+1) \\ n_2(t+1) \\ n_3(t+1) \\ \vdots \\ n_k(t+1) \end{bmatrix},$$

with $n_k(t+1)$ is the number of females in the *k*-th age class at time (*t* + 1), and the number of females in the first age class at time (*t* + 1) comprises those born between time *t* and (*t* + 1), defined as:

$$n_1(t+1) = a_1 n_1(t) + a_2 n_2(t) + \cdots + a_k n_k(t). \tag{3}$$

The number of females in age class (*i* + 1) at time *t* + 1 consists of those who were in class *i* at time *t* and survived until *t* + 1, expressed as:

$$n_{i+1}(t+1) = b_i n_i(t), \qquad i = 1, 2, 3, \ldots, k-1. \tag{4}$$

Using the Leslie matrix, equation (3) and (4) can be expressed as a population growth model:

$$\begin{bmatrix} n_1(t+1) \\ n_2(t+1) \\ n_3(t+1) \\ \vdots \\ n_k(t+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{k-1} & 0 \end{bmatrix} \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ \vdots \\ n_k(t) \end{bmatrix},$$

or more concisely:

$$\mathbf{n}(t+1) = \mathbf{L}\mathbf{n}(t). \tag{5}$$

To predict the population in the next *p* periods, equation (5) is iterated as follows:

$$\begin{aligned} \mathbf{n}(t+1) &= \mathbf{L}\mathbf{n}(t) \\ \mathbf{n}(t+2) &= \mathbf{L}\mathbf{n}(t+1) = \mathbf{L}\mathbf{L}\mathbf{n}(t+1) = \mathbf{L}^2\mathbf{n}(t) \\ \mathbf{n}(t+3) &= \mathbf{L}\mathbf{n}(t+2) = \mathbf{L}\mathbf{L}^2\mathbf{n}(t+1) = \mathbf{L}^3\mathbf{n}(t) \\ &\vdots \\ \mathbf{n}(t+p) &= \mathbf{L}\mathbf{n}\big(t+(p-1)\big) = \mathbf{L}\mathbf{L}^{p-1}\mathbf{n}\big(t+(p-1)\big) = \mathbf{L}^p\mathbf{n}(t) \end{aligned}$$

Thus, the population distribution vector p periods into the future is given by:

$$\mathbf{n}(t+p) = \mathbf{L}^p\mathbf{n}(t). \tag{6}$$

If the initial age distribution and the Leslie matrix **L** are known, the female population distribution over the next *p* periods can be determined.

The eigenvalues of the Leslie matrix indicate the population growth rate—whether it tends to increase, decrease, or remain stable (Pratama et al., 2013). The eigenvalues are the roots of the characteristic polynomial of the Leslie matrix, expressed as:

$$p(\lambda) = |\lambda\mathbf{I} - \mathbf{L}| = 0 \tag{7}$$

Theorem 1 (Anton & Rorres, 2014). *A Leslie matrix has a unique positive eigenvalue λ_1, with multiplicity 1, and it has a corresponding eigenvector $\boldsymbol{x}_1$ whose entries are all positive*.

Definition 1 – Dominant Eigenvalue (Anton & Rorres, 2014). *Given eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of an $n \times n$ matrix A, λ_1 is called the dominant eigenvalue of A if $|\lambda_1| > |\lambda_i|$ for $i = 2, 3, \dots, n$.*

The population growth rate can be determined by the dominant eigenvalue of the Leslie matrix:

a. If $\lambda_1 = 1$, the population is stable;
b. If $\lambda_1 > 1$, the population tends to grow;
c. If $\lambda_1 < 1$, the population tends to decline.

2.2. Harvesting Animal Populations Following the Leslie Matrix Model

Animal population harvesting can be carried out when the population growth rate shows an increasing trend, indicated by the dominant eigenvalue of the Leslie matrix being greater than one (Vindenes et al., 2021). In this context, harvesting refers to the removal or extraction of a portion of the individuals from the population for other purposes, such as consumption, production, or conservation. This process is typically performed at the end of each growth period.

Let h_i, for $i = 1, 2, \dots, n$ represent the proportion of individuals harvested from the i-th age class. The harvesting matrix can then be defined as a diagonal matrix of size $n \times n$ as follows:

$$\mathbf{H} = \begin{bmatrix} h_1 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 0 & \cdots & 0 \\ 0 & 0 & h_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_n \end{bmatrix}; 0 \le h_i \le 1.$$

The vector product **HLx** denotes the number of female individuals harvested from each age class, where x is the age distribution vector of the population. To ensure sustainable harvesting, a balance condition must be satisfied such that the population remains constant over time despite the harvesting process. This condition can be expressed by the following equation:

$$(\mathbf{I} - \mathbf{H})\boldsymbol{L}\mathbf{x} = \mathbf{x} \tag{8}$$

3. Results and Discussion

3.1. Leslie Matrix for Female Cattle Population Growth

In this study, the female dairy cattle population was divided into 3 age classes, and the female beef cattle population was divided into 11 age classes. Based on the data, the fertility rates (a_i) d and survival rates (b_i) of the population were calculated as follows:

Table 1: Fertility and Survival Rates of Dairy Cattle

Age Class	Fertility Rate (a_i)	Survival Rate (b_i)
1	0	0.491
2	0	5.761
3	0.316	-

Table 2: Fertility Rate and Survival Rate of Beef Cattle

Age Class	Fertility Rate (a_i)	Survival Rate (b_i)
1	0	0.933
2	0	0.929
3	0.292	0.917
4	0.370	0.963
5	0.400	1
6	0.438	1
7	0.417	0.972
8	0.323	1
9	0.208	0.958
10	0.167	0.833
11	0	-

Based on the data presented in Table 1 and Table 2, Leslie matrices were constructed. For the population of female dairy cattle, which consists of three age classes, the Leslie matrix is of size 3×3. Sela Meanwhile, the female beef cattle population consists of eleven age classes, resulting in an 11×11 Leslie matrix. The Leslie matrices for both populations are as follows:

$$\mathbf{L_1} = \begin{bmatrix} 0 & 0 & 0.316 \\ 0.491 & 0 & 0 \\ 0 & 5.761 & 0 \end{bmatrix} \text{ and}$$

$$\mathbf{L_2} = \begin{bmatrix} 0 & 0 & 0.292 & 0.37 & 0.4 & 0.438 & 0.417 & 0.323 & 0.208 & 0.167 & 0 \\ 0.933 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.929 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.917 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.963 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.972 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.958 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.833 & 0 \end{bmatrix}$$

The initial population $n(t)$ ad corresponds to the population data from the year 2021, and the time step for each growth period p is one year. Thus, the Leslie matrix population growth model can be expressed as:

$$\mathbf{n}(2021 + p) = \mathbf{L}^p \mathbf{n}(2021). \tag{9}$$

Based on this model, the projected female dairy cattle population at BBPTU-HPT Baturraden for the years 2022–2025 is presented in Table 3 and Table 4.

Table 3: Predicted Population of Female Dairy Cattle

Age Class	Year			
	2022	2023	2024	2025
1	197	198	189	176
2	104	97	97	93
3	628	600	558	561
Total	929	895	844	830

Table 4: Predicted Population of Female Beef Cattle

Age Class	Year			
	2022	2023	2024	2025
1	75	82	84	92
2	42	70	76	78
3	52	39	65	76
4	22	48	36	34
5	26	21	46	46
6	30	26	21	21
7	32	30	29	25
8	35	31	31	29
9	31	35	33	30
10	23	30	33	30
11	10	19	25	28
Total	378	431	472	514

Based on Table 3, the predicted population of female dairy cattle from 2022 to 2025 shows a consistent decline. In contrast, Table 4 demonstrates that the predicted population of female beef cattle is steadily increasing during the same period.

Subsequently, the population growth rate of female cattle is determined by calculating the eigenvalues of the corresponding Leslie matrix. For the female dairy cattle population, the growth rate is obtained by solving the characteristic polynomial:

$$p(\lambda) = |\lambda\mathbf{I} - \mathbf{L}| = 0$$

$$\begin{vmatrix} \lambda & 0 & -0.316 \\ -0.491 & \lambda & 0 \\ 0 & -5.761 & \lambda \end{vmatrix} = 0$$

Using Maple13 software, the eigenvalues of the Leslie matrix $\mathbf{L}$ are obtained as follows:

$\lambda_1 = 0{,}9632865209,\quad \lambda_2 = -0{,}48216432605 + 0{,}8342305982\,i,\quad \lambda_3 = -0{,}48216432605 - 0{,}8342305982\,i.$

The dominant eigenvalue is $\lambda_1 = 0{,}963$. Since the dominant eigenvalue $\lambda_1 < 1$, it indicates that the female dairy cattle population is projected to continue decreasing over time.

For the female beef cattle population, the growth rate is determined similarly by solving:

$$p(\lambda) = |\lambda\mathbf{I} - \mathbf{L}| = 0$$

$$\begin{vmatrix} \lambda & 0 & -0.292 & -0.370 & -0.400 & -0.438 & -0.417 & -0.323 & -0.208 & -0.167 & 0 \\ -0.933 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.929 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.917 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.963 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.972 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.958 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.833 & \lambda \end{vmatrix} = 0$$

Using Maple13, the eigenvalues of the Leslie matrix for beef cattle are calculated as:

$\lambda_1 = 1.129105741,$
$\lambda_2 = 0.5461674280 + 0.6110252640i,$
$\lambda_3 = 0.5461674280 - 0.6110252640i,$
$\lambda_4 = 0.1445005802 + 0.7455336287i,$
$\lambda_5 = 0.1445005802 - 0.7455336287i,$
$\lambda_6 = -0.2648647334 + 0.7487889666i,$
$\lambda_7 = -0.2648647334 - 0.7487889666i,$
$\lambda_8 = -0.6158322836 + 0.4435098176i,$
$\lambda_9 = -0.6158322836 - 0.4435098176i,$
$\lambda_{10} = -0.7490477237,$
$\lambda_{11} = 0.$

The dominant eigenvalue is $\lambda_1 = 1{,}129$. Since $\lambda_1 > 1$ this indicates that the female beef cattle population is expected to grow over time.

3.2. Harvesting Animal Populations Following the Leslie Matrix Model

Harvesting of livestock can be implemented if the population growth rate tends to increase, or in other words, when the dominant eigenvalue of the Leslie matrix is greater than one. As presented in Section 3.1, the only population with an increasing growth trend is the female beef cattle population. Therefore, the harvesting strategy will focus solely on this group.

Uniform harvesting can be applied when animals are removed randomly, without consideration of their age group. Under this assumption, all age classes are subject to the same harvesting proportion, such that $h = h_1 = h_2 = \cdots = h_n$. The uniform harvesting proportion can be calculated using the formula:

$$h = 1 - \left(\frac{1}{\lambda_1}\right)$$
$$= 1 - \frac{1}{1.129} = 0.114$$

Based on this calculation, the uniform harvesting proportion is 0.114, which means that 11.4% of the female beef cattle population can be harvested from each age class.

Selective harvesting of the youngest age group involves only the first age class. Prior to calculating the harvesting proportion for the youngest group, it is necessary to determine the net reproductive rate of the population using the formula:

$$R = a_1 + a_2b_1 + a_3b_1b_2 + \cdots + a_{11}b_1b_2 \ldots b_{10}$$
$$R = 0 + 0(0.933) + 0.292(0.867) + 0.370(0.795) + 0.400(0.765) + 0.438(0.765) + 0.417(0.765)$$
$$+0.323(0.744) + 0.208(0.744) + 0.167(0.713) + 0(0.594)$$
$$R = 2.021$$

The net reproductive rate is thus calculated to be $R = 2.021$. Consequently, the harvesting proportion for the youngest age class is given by:

$$h = 1 - \frac{1}{2.021}$$
$$h = 1 - \frac{1}{2.021} = 0.505$$

This means that 50.5% of the individuals in the first age class can be harvested. If the total number in this class is 73 individuals, then the number that can be sustainably harvested is approximately 37 individuals.

4. Conclussion

Based on the results and discussion, it can be concluded that the application of the Leslie matrix model in predicting and analyzing the growth rate of the female cattle population indicates that the population of dairy female cattle is projected to decrease from 2022 to 2025, with a declining growth rate as reflected by an eigenvalue of 0.963. In contrast, the population of beef female cattle is expected to increase during the same period, with a growing trend shown by an eigenvalue of 1.129. Therefore, harvesting is only carried out on the beef cattle population, which has a positive growth rate, with a uniform harvesting proportion of 11.4% across all age classes and a harvesting proportion of 50.5% for the youngest age class. Future research is recommended to include population prediction for extended periods under sustainable harvesting scenarios, as well as to explore harvesting strategies for other age classes.

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