



Accuracy of the Jian-Yang-Jiang-Liu-Liu Spectral Conjugate Gradient Method in Estimating Extended Exponential Weibull Parameters

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Abstract

Life is filled with uncertainty and risk. The analysis of lifetime is needed to be a tool that can manage uncertainty. Lifetime is defined as data that contains the time until the occurrence of an event. Based on its definition, lifetime data is like Hazard rate data or mortality data because mortality data can be defined as data that contains the probability of an object surviving until that moment per unit time interval. The analysis of mortality data aims to model the distribution of time to event and/or the determinants of time to event. One of the distribution models that can be used to analyze mortality data is Weibull distribution. However, the Weibull distribution is not very suitable for modeling the more complex versions of data. Therefore, an extension of the Weibull distribution that is more flexible in modeling data is used, namely the Extended Exponential Weibull (ExEW) distribution. The ExEW distribution has four parameters whose estimation can be calculated using the maximum likelihood estimation (MLE) method. However, parameters estimated with MLE are often too difficult to calculate analytically, hence the use of optimization methods. One of the optimization methods that can be used to determine the estimated parameters of the ExEW distribution is the conjugate gradient method. To date, many conjugate gradient methods have been developed, including the Liu-Feng-Zou (LFZ) spectral conjugate gradient method and the Jian-Yang-Jiang-Liu-Liu (JYJLL) spectral conjugate gradient method. Previous research suggests that the JYJLL spectral conjugate gradient method has more efficient computational performance than the LFZ spectral conjugate gradient method. Through data simulation, this study provides results that the JYJLL spectral conjugate gradient conjugate method has better accuracy than the LFZ spectral conjugate gradient method in parameter estimation of the ExEW distribution. In addition, the ExEW distribution is the most suitable distribution in modeling various forms of Hazard rate data compared to the Weibull and exponential distributions.

Keywords: Exponential distribution, maximum likelihood, mortality data, numerical optimization, Weibull distribution.

1. Introduction

Lifetime data are defined as data that contain the time until a particular event occurs (Ahmad et al., 2020). Based on this definition, lifetime data are like hazard rate or mortality data. The hazard rate is defined as the probability that an object survives up to a certain time per unit of time interval (Kumar, 2023). Therefore, mortality data can be interpreted as data representing the probability that an object survives up to a certain point in time per unit interval.

Analyzing lifetime data is a powerful tool for prediction, especially when used to model probability distributions. In the context of mortality data, distribution modeling can provide insights into the likelihood of component failure or individual death at specific points in time. As such, the hazard rate becomes essential for observing mortality patterns. When predicting data over a specific time frame, it is necessary to determine the best-fitting method to identify the most appropriate probability density function (PDF) and estimate its parameters to plot the hazard rate (Calixto, 2016).

Various probability models exist to model lifetime data, such as the Weibull and exponential distributions (Nasiru et al., 2019). However, in many cases, these models are no longer optimal or adequate. Given the diverse nature of mortality data, the selection and development of appropriate distributions have become a key focus in research to enable accurate data analysis. Hence, efforts continue to develop new distributions that can model lifetime data with greater accuracy (Mastor et al., 2022).

The Weibull distribution is among the most well-known models for modeling lifetime data such as mortality (Matsushita et al., 1992). Its primary advantage lies in its flexibility to mimic other distributions, such as the normal or

exponential distribution, and its ease of interpretation through its PDF and cumulative distribution function. Its parameters are easily determined and flexible, supporting the use of Weibull distribution in modeling various datasets. Furthermore, its ability to model skewed density functions makes it particularly suitable for modeling monotonic hazard rates (He et al., 2020).

Aside from the Weibull distribution, other models for lifetime data include the exponential, linear failure rate (LFR), and Rayleigh distributions (Cordeiro et al., 2011). However, these models are typically suitable only for specific forms of hazard functions. Thus, the Weibull distribution is often used to generalize such distributions, allowing it to accommodate constant, increasing, or decreasing hazard rates. As the field progresses, numerous new distributions have been proposed to address various types of hazard shapes. Motivated by this, Cordeiro et al. (2014) introduced the Exponential Weibull (EW) distribution, developed by generalizing the random variables from the Weibull and exponential distributions. This model offers enhanced flexibility, allowing it to better accommodate more complex datasets.

It is important to note that the exponential distribution is commonly used to model the time between events in a Poisson process, while the Weibull distribution is used to analyze time-to-failure data. Accordingly, the Exponential Weibull random variable can be interpreted as the time to the nearest failure (Cordeiro et al., 2014).

Advancements in the Weibull distribution have inspired the use of various new techniques, including parameter induction methods. Parameter induction introduces one or more shape parameters into an existing distribution, enhancing its flexibility—particularly in handling more complex hazard shapes. Following this approach, this study explores a new probability distribution—the Extended Exponential Weibull (ExEW) distribution—as a modified form of the Exponential Weibull distribution. The approach uses the Lehmann Alternative Type II (LA-II) method, one of the key methods for developing exponential-type distribution families. LA-II is applied to the ExEW distribution to improve its ability to handle more complex hazard rate behaviors (Tahir & Nadarajah, 2015).

By using the LA-II parameter induction method, the ExEW distribution can accommodate monotonic, non-monotonic, and constant hazard rates. While the traditional Weibull distribution effectively models skewed densities, its generalization—the ExEW distribution—offers greater adaptability, particularly in representing a wider range of kurtosis (peakedness) levels. This advantage will be demonstrated by modeling a real-world dataset, namely airplane windshield failure rate data.

The dataset will be fitted to the ExEW distribution to estimate its parameters. Optimal parameter estimation is essential for accurately modeling the available data. One of the most used techniques for parameter estimation is Maximum Likelihood Estimation (MLE). As the name suggests, MLE maximizes the likelihood function to estimate parameters based on the observed data. The likelihood function represents the joint probability of all data given the parameters to be estimated (Klugman et al., 2012).

In practice, analytically solving the MLE equations for parameter estimation can be challenging. Thus, numerical methods are often employed to support MLE computations. One such numerical method is the conjugate gradient method, widely used for large-scale optimization problems due to its low iteration count, fast convergence, and minimal memory requirements (Hestenes & Stiefel, 1952). Among its variants is the spectral conjugate gradient method, first proposed by Birgin & Martinez (2001), which generalizes traditional conjugate gradient approaches. The primary difference lies in its search direction, which includes an additional spectral parameter. This spectral parameter enhances flexibility and has been shown to produce more decreasing values per iteration compared to standard conjugate gradient methods.

Several forms of the spectral conjugate gradient method exist, including the Liu-Feng-Zou (LFZ) conjugate gradient method and the Jian-Yang-Jiang-Liu-Liu (JYJLL) conjugate gradient method. The LFZ method, introduced by Liu, Feng, & Zou (2019), was among the earliest spectral conjugate gradient methods, designed to simplify the selection of conjugate direction coefficients. Each coefficient is uniquely defined by iteration based on the method used. The LFZ method's strength lies in its effectiveness in simulating various other conjugate gradient methods and its consistently descending search direction regardless of the line search technique. However, the LFZ method remains somewhat susceptible to errors at each iteration.

To address these limitations, Jian et al. (2020) proposed the JYJLL spectral conjugate gradient method as a safer and more efficient alternative. Its key advantage lies in the addition of a spectral parameter that reduces the number of iterations, thus accelerating the computation process.

In previous studies, comparisons have been made between the efficiency of the LFZ and JYJLL methods, demonstrating that the JYJLL method offers superior computational performance. However, those studies did not investigate the accuracy of the two methods. Therefore, this research focuses on analyzing the accuracy performance of the JYJLL and LFZ spectral conjugate gradient methods in supporting MLE for parameter estimation in the ExEW distribution.

The analysis is carried out through simulation using datasets from three distributions, and accuracy is measured using Root Mean Square Error (RMSE). In addition, the Kolmogorov–Smirnov test is applied to evaluate the

goodness-of-fit of the ExEW, Weibull, and exponential distributions to real-world mortality data (airplane windshield failure rate). Finally, the best-fitting model among the three competing distributions is selected using model selection criteria: Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC).

2. ExEW Distribution and MLEs of the Parameters

The Extended Exponential Weibull (ExEW) distribution was first discovered by Mastor et al. (2022). The ExEW distribution is an extension of the exponential Weibull using the Lehmann alternative II (LA-2) parameter induction technique. Inducing one or more shape parameters to a family of distributions helps the exponential distribution family to be more flexible in its use. The ExEW distribution, which is a generalization of the exponential Weibull distribution, is formulated using the LA-2 approach with the addition of one shape parameter.

Suppose a random variable $X \sim \text{ExEW}(a, b, c, \alpha)$. The cumulative distribution function of the ExEW distribution is constructed using the Lehmann II technique on the cumulative distribution function ExEW as follows:

$$F_{X \sim \text{ExEW}}(x) = 1 - [1 - (1 - e^{-(ax+bx^c)})]^\alpha = 1 - [1 - F_{X \sim \text{EW}}(x)]^\alpha = 1 - [e^{-(ax+bx^c)}]^\alpha, x > 0,$$

where $\alpha > 0$, $a > 0$, and $c > 0$ are shape parameters and $b > 0$ is the scale parameter. Then, the probability density function of the ExEW distribution is constructed by differentiating the cumulative distribution function of ExEW with respect to x as follows:

$$f_X(x) = \frac{d}{dx} [1 - e^{-(ax+bx^c)}]^\alpha = \alpha [(a + bcx^{c-1})e^{-(ax+bx^c)}] [e^{-(ax+bx^c)}]^{\alpha-1}, x > 0.$$

Meanwhile, the survival function and hazard function are respectively defined as follows:

$$S(x) = 1 - F(x) = [e^{-(ax+bx^c)}]^\alpha, h(x) = \frac{f(x)}{S(x)} = \frac{\alpha [(a + bcx^{c-1})e^{-(ax+bx^c)}]}{[e^{-(ax+bx^c)}]^\alpha}.$$

In Figures 1 and 2, the density functions (PDF) and hazard functions (HF) are presented for several values of the parameter.

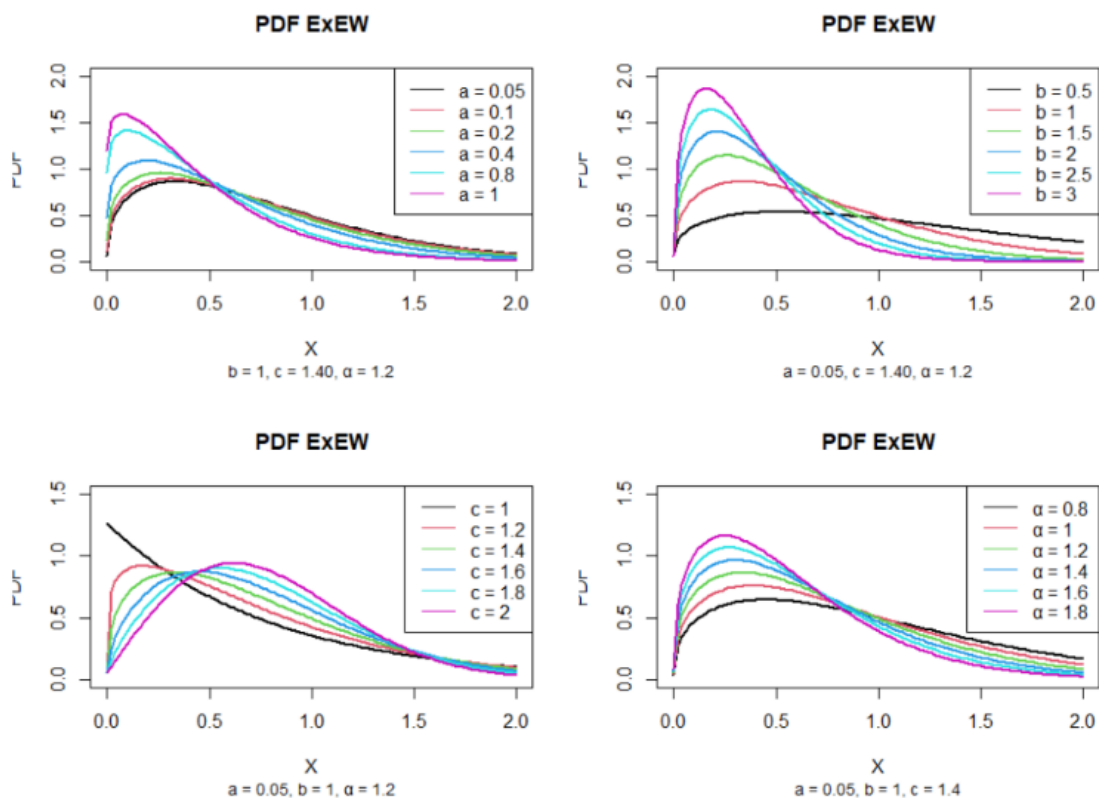


Figure 1: Density of the ExEW distribution

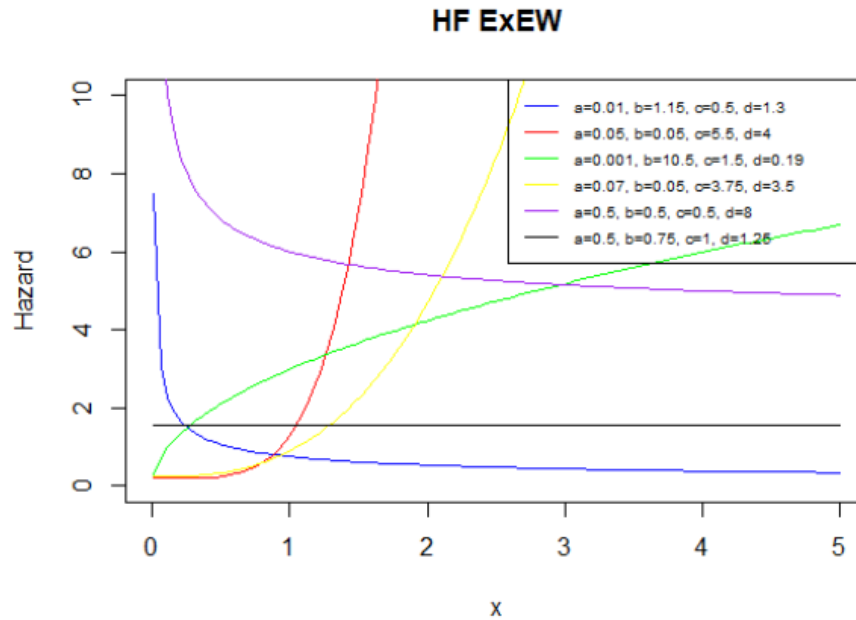


Figure 2: Hazard of the ExEW distribution

Based on the selection of parameters in Figure 1, the shape of the ExEW probability density function varies, namely symmetric, asymmetric, unimodal, J, and reversed J forms. Note that the smaller the values of parameters a, b , and α , the more left the curve is skewed and the steeper the curve formed. Meanwhile, the larger the value of parameter c , the more left the curve is skewed. The plot form of the hazard function can be increasing, decreasing, constant, J, and reversed J.

The estimation of ExEW distribution parameters is done using the Maximum Likelihood Estimation (MLE) method. Suppose $x_1, x_2, \dots, x_i$ are the observed values of random samples $X_1, X_2, \dots, X_i$ which are distributed ExEW with parameter vector $\phi = (a, b, c, \alpha)$ and probability density function $f(x; \phi)$. The likelihood and log-likelihood functions are obtained as follows:

$$L(\phi) = \alpha^n e^{-\sum_{i=1}^n (ax_i + bx_i^c)} \cdot \prod_{i=1}^n (a + bcx_i^{(c-1)}) \cdot \prod_{i=1}^n [e^{-(ax_i + bx_i^c)}]^{(\alpha-1)}.$$

Next, the likelihood function is expressed in natural logarithm form to find the values a, b, c , and α that maximize the likelihood function. The $\ln$ likelihood function is obtained as follows:

$$l(\phi) = n \ln \alpha - \alpha \sum_{i=1}^n (ax_i + bx_i^c) + \sum_{i=1}^n \ln (a + bcx_i^{(c-1)}). \quad (1)$$

Then, the estimated values for parameters a, b, c , and α are obtained by maximizing the form $l(\phi)$. These values are obtained by finding the first partial derivative of $l(\phi)$ with respect to a, b, c , and α which are then equated to zero. As follows:

$$\begin{aligned} \frac{\partial l(\phi)}{\partial a} &= -\alpha \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{1}{a + bcx_i^{(c-1)}} = 0. \\ \frac{\partial l(\phi)}{\partial b} &= -\alpha \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{cx_i^{(c-1)}}{a + bcx_i^{(c-1)}} = 0. \\ \frac{\partial l(\phi)}{\partial c} &= -\alpha \sum_{i=1}^n bx_i^c \cdot \ln x_i + b \sum_{i=1}^n \left(\frac{cx_i^{(c-1)} \cdot \ln x_i + x_i^{(c-1)}}{ax_i + bcx_i^{(c-1)}} \right) = 0. \end{aligned}$$

$$\frac{\partial l(\phi)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n (ax_i + bx_i^c) = 0.$$

The partial derivatives of the four parameters are not explicit, so it is necessary to use a numerical method in estimating the parameters. The numerical method used is the JYJLL spectral conjugate gradient method which is then compared with the LFZ spectral gradient conjugate method.

3. The Spectral Conjugate Gradient Method for Estimating ExEW Parameters

The spectral conjugate gradient method is one of the gradient-based iterative optimization methods that can be used to solve nonlinear optimization problems. This method uses the iterative formula:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, 2, \dots \quad (2)$$

where x_k is the point at the k th iteration, x_{k+1} denotes the $(k+1)$ th iteration, $\alpha_k > 0$ denotes the step length and can be calculated using various methods (exact or inexact line search), and d_k denotes the search direction as follows:

$$d_k = \begin{cases} -g_k, & k = 0, \\ -\theta_k g_k + \beta_k d_{k-1}, & k \geq 1, \end{cases} \quad (3)$$

where $g_k = g(x_k) = \nabla f(x_k)$ is the gradient of the objective function at the point x_k , θ_k is the spectral scalar, and β_k is a scalar representing the conjugate direction coefficient, determined by various types of conjugate gradient methods. The point x_{k+1} is updated iteratively so that, in each iteration, the value of the objective function is lower than in the previous one (in the case of minimization). This process continues using new approximation points until a point x^* is obtained that minimizes the objective function and/or until the stopping criterion is satisfied.

Until now, many researchers have proposed various spectral conjugate gradient methods. Among them is the LFZ and JYJLL spectral conjugate gradient methods.

The LFZ spectral gradient conjugate method was proposed by Liu et al. (2019) with the aim of creating a spectral search direction that satisfies the sufficient descent condition:

$$g_k^T d_k \leq -C \|g_k\|^2, \quad k = 0, 1, 2, \dots$$

for $C = 1$. The form of the parameter θ_k in the search direction (3) for the LFZ method is defined as follows:

$$\theta_k^{LFZ} = -\frac{g_{k-1}^T d_{k-1}}{\|g_{k-1}\|^2} + \beta_k \frac{g_k^T d_{k-1}}{\|g_k\|^2}$$

and β_k is given by

$$\beta_k^{LFZ} = \max \left\{ \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \min \left\{ \max \left\{ 0, \frac{g_k^T (g_k - g_{k-1})}{\|g_{k-1}\|^2} \right\}, \frac{\|g_k\|^2}{\|g_{k-1}\|^2} \right\} \right\}.$$

The LFZ method was proposed for its ability to guarantee the descent property without relying on any line search rule, offering computational efficiency and strong performance for quadratic or near-quadratic problems. However, it has limitations in maintaining search direction stability and achieving fast convergence in more complex non-quadratic problems. To address these issues, the JYJLL method was developed by modifying the spectral parameter through a combination of Improve-Fletcher-Reeves and Improve-Dai-Yuan (IFR-IDY) conjugate gradient methods (masukkan), thereby producing more stable search directions, ensuring a more consistent descent property, and improving convergence across various optimization problems. In the JYJLL spectral conjugate gradient method, the parameter θ_k and β_k used in the search direction in (3) is defined as follows, respectively.

$$\theta_k^{JYJLL} = 1 + \frac{|g_k^T d_{k-1}|}{-g_{k-1}^T d_{k-1}}, \quad (4)$$

$$\beta_k^{JYJLL} = \frac{\|g_k\|^2 - \frac{(g_k^T d_{k-1})^2}{\|d_{k-1}\|^2}}{\max\{\|g_{k-1}\|^2, d_{k-1}^T (g_k - g_{k-1})\}}. \quad (5)$$

After explaining the JYLL spectral conjugate gradient method, the following are the steps for using the JYJLL conjugate gradient method to determine the maximum likelihood parameter estimates of the ExEW distribution:

- **Step 0: Initialization**

Set the starting point starting point $x_0 = (a_0, b_0, c_0, \alpha_0)$ and objective function $f(a, b, c, \alpha) = -l(a, b, c, \alpha)$, where l is log-likelihood function as in (1).

- **Step 1: Calculate Gradient**

The gradient of the objective function is given by

$$\nabla f(a, b, c, \alpha) = -\nabla l(a, b, c, \alpha) = - \begin{pmatrix} \frac{\partial l(a, b, c, \alpha)}{\partial a} \\ \frac{\partial l(a, b, c, \alpha)}{\partial b} \\ \frac{\partial l(a, b, c, \alpha)}{\partial c} \\ \frac{\partial l(a, b, c, \alpha)}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} -\alpha \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{1}{a + bcx_i^{(c-1)}} \\ -\alpha \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{cx_i^{(c-1)}}{a + bcx_i^{(c-1)}} \\ -\alpha \sum_{i=1}^n bx_i^c \cdot \ln x_i + b \sum_{i=1}^n \left(\frac{cx_i^{(c-1)} \cdot \ln x_i + x_i^{(c-1)}}{ax_i + bcx_i^{(c-1)}} \right) \\ \frac{n}{\alpha} - \sum_{i=1}^n (ax_i + bx_i^c) \end{pmatrix}.$$

If $\|\nabla f(a, b, c, \alpha)\| \leq 10^{-6}$, then stop and x_k is the optimal vector. If not, then proceed to the next step.

- **Step 2: Determining Search Direction**

- If $k = 0$, then $d_0 = -g_0 = -\nabla f(a, b, c, \alpha) = \nabla l(a, b, c, \alpha)$.
- If $k > 0$, then $d_k = -\theta_k^{JYJLL} g_k + \beta_k^{JYJLL} d_{k-1} = \theta_k^{JYJLL} \nabla l(a, b, c, \alpha) + \beta_k^{JYJLL} d_{k-1}$, where θ_k^{JYJLL} and β_k^{JYJLL} are given in (4) and (5), respectively.

- **Step 3: Determining Step Length**

Determine α_k using golden section search to minimize $f(x_k + \alpha_k d_k)$.

- **Step 4: Update Point**

Update the points using the iterative formula (2). Increase the indeks $k = k + 1$, then return to step 2 until the stopping criterion is reached.

By following the steps outlined above, the JYJLL conjugate spectral gradient method can be utilized to help calculate the maximum likelihood values of the ExEW distribution parameters. This approach can also be applied in a similar manner to the LFZ conjugate spectral gradient method in estimating the maximum likelihood parameters of the ExEW distribution.

4. Results and Discussion

To evaluate the effectiveness and accuracy of the proposed parameter estimation methods, this study employs a two-stage approach: simulation and real data application. The simulation study is conducted to assess the performance of the spectral conjugate gradient methods—specifically the JYJLL spectral conjugate gradient method—in a controlled environment where the true parameters are known. This enables a precise comparison of the accuracy and stability of both methods using Root Mean Square Error (RMSE) as the performance metric.

Following the simulation analysis, the methods are further validated through application to a real-world dataset: the airplane windshield failure rate data. This empirical analysis aims to demonstrate the practical applicability of the Extended Exponential Weibull (ExEW) distribution in modeling actual lifetime data and to assess the model fit using standard statistical criteria such as the Kolmogorov–Smirnov (K–S) test, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC).

The simulation begins by generating random data using RStudio software which is then substituted into the ExEW log-likelihood function. Then, the log-likelihood function is maximized (or minimized the negative log-likelihood function) with the same software using both the JYJLL and LFZ conjugate gradient methods. The parameter estimation results from both methods are compared using Rooted Mean Square Error (RMSE).

The random data generation process is carried out by following the guidelines from the journal "New Extension of Burr Type X Distribution Properties with Application" by Khaleel, et al. (2018). This process involves the inverse CDF form of the ExEW distribution and assumes that the parameter values can be generated randomly with an ExEW distribution. In this simulation, the parameter pairs used are obtained from the journal "The Extended Exponential

Weibull Distribution: Properties, Inference, and Applications to Real-Life Data" by Mastor et al. (2022). There are two pairs used, namely set 1: $(a, b, c, \alpha) = (1, 1, 1.5, 0.12)$ and set 2: $(a, b, c, \alpha) = (1, 1, 1.5, 0.14)$. In this simulation, the sample sizes used are 25, 50, 75, 100, 150, 200, and 300.

After generating the simulation data, the data are substituted into the log-likelihood function of the ExEW distribution to estimate the parameters a, b, c , and α . With the help of Rstudio software, the log-likelihood function is optimized using two conjugate spectral gradient methods, namely JYJLL and LFZ. The goal is to obtain estimates of the MLE parameters denoted as $\hat{a}, \hat{b}, \hat{c}$, and $\hat{\alpha}$.

4.1.1. Parameter Estimation Results

RMSE is an evaluation criterion used in this study to compare the results of parameter estimation from simulations. Since parameter estimation is performed 500 times, then for each conjugate spectral gradient method, estimation replication is performed 500 times.

Tables 1 and 2 present the average simulation results from 500 iterations of estimation for the set 1 along with their RMSE with the JYJLL and LFZ spectral gradient conjugate methods and Figure 5 shows the comparison.

Table 1: Simulation results of set 1 with the JYJLL spectral gradient conjugate method for 500 iterations

n	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$RMSE \hat{a}$	$RMSE \hat{b}$	$RMSE \hat{c}$	$RMSE \hat{\alpha}$
25	0.441	0.3619	1.5817	0.352	0.6726	0.7209	0.6797	0.2577
50	0.4236	0.4145	1.5659	0.3347	0.6714	0.7151	0.537	0.2382
75	0.4185	0.4392	1.5477	0.3286	0.6873	0.6746	0.454	0.3052
100	0.4363	0.4533	1.6051	0.3074	0.6391	0.789	0.4029	0.2072
150	0.4105	0.4894	1.5619	0.3074	0.7348	0.7094	0.3541	0.2137
200	0.4349	0.4341	1.5879	0.2969	0.6858	0.6265	0.3413	0.1896
300	0.4331	0.4612	1.5724	0.2877	0.6596	0.6283	0.2558	0.1768
Average					0.6787	0.6948	0.4321	0.2269

Table 2: Simulation results of set 1 with the LFZ spectral gradient conjugate method for 500 iterations

n	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$RMSE \hat{a}$	$RMSE \hat{b}$	$RMSE \hat{c}$	$RMSE \hat{\alpha}$
25	0.6578	0.928	2.0362	0.2398	0.9098	1.7188	1.5647	0.1971
50	0.6284	0.9274	1.9302	0.2466	0.5905	1.4258	1.281	0.4038
75	0.6568	1.2605	1.7748	0.2607	0.6061	3.0262	0.9656	0.4993
100	0.6656	1.0197	1.7667	0.2321	0.5452	1.4918	0.8208	0.368
150	0.7074	1.1741	1.6583	0.2151	0.8312	1.9511	0.5947	0.3636
200	0.7597	1.0289	1.6491	0.1821	1.1438	3.3023	0.5799	0.1242
300	0.7104	1.2401	1.5673	0.1941	0.8633	1.9203	0.5125	0.2926
Average					0.7843	1.8338	0.9028	0.3212

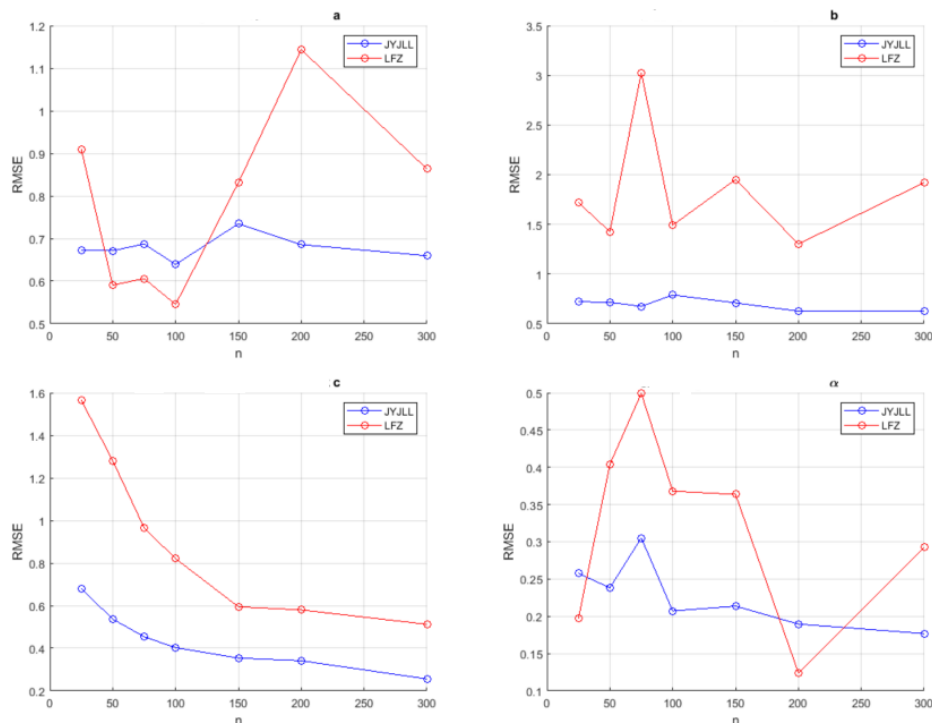


Figure 5: RMSE per Parameter of the JYJLL (blue) and LFZ (red) Conjugate Spectral Gradient Methods for Set 1

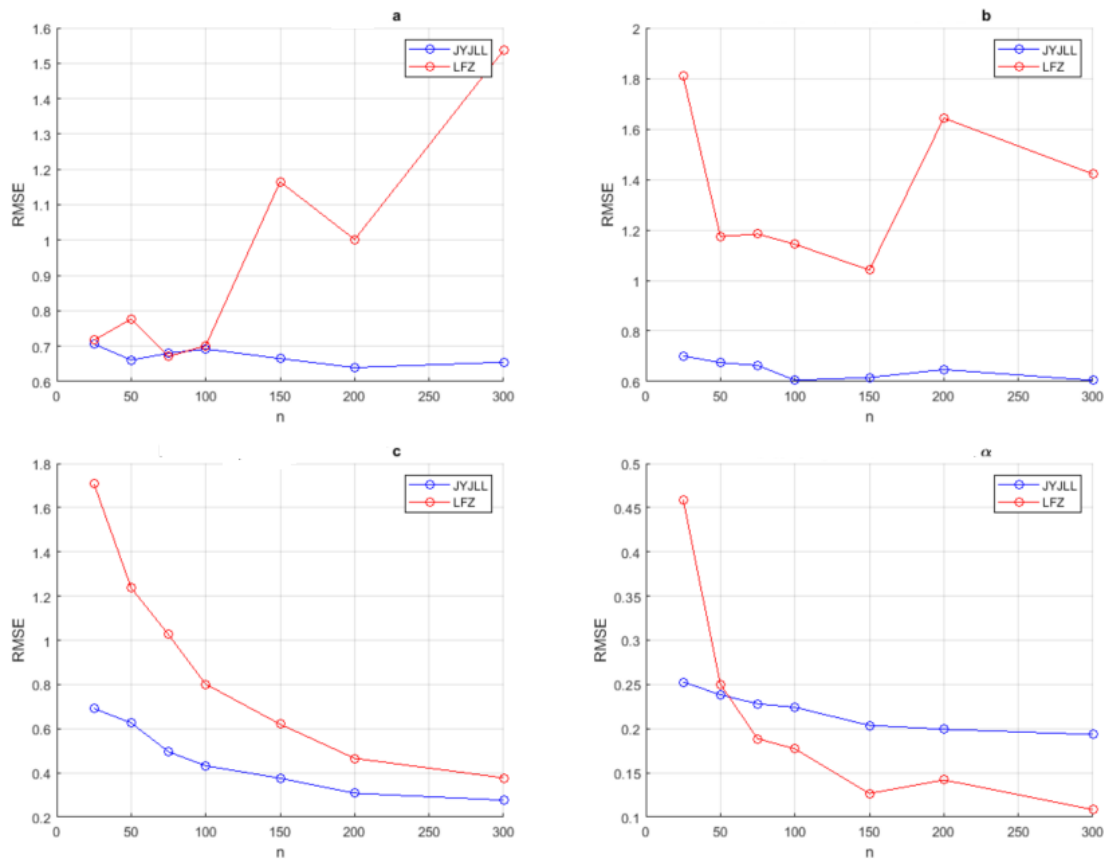
Tables 3 and 4 present the average simulation results from 500 iterations of estimation for the set 2 along with their RMSE with the JYJLL and LFZ spectral gradient conjugate methods and Figure 6 shows the comparison.

Table 3: Simulation results of set 2 with the JYJLL spectral gradient conjugate method for 500 iterations

n	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$RMSE \hat{a}$	$RMSE \hat{b}$	$RMSE \hat{c}$	$RMSE \hat{\alpha}$
25	0.3948	0.4398	1.7090	0.3652	0.7062	0.7008	0.6908	0.2527
50	0.4291	0.4274	1.7532	0.3533	0.6608	0.6750	0.6260	0.2383
75	0.3996	0.4612	1.6898	0.3462	0.6800	0.6637	0.4951	0.2281
100	0.3873	0.4868	1.6423	0.3436	0.6920	0.6049	0.4318	0.2241
150	0.4289	0.4937	1.6248	0.3247	0.6649	0.6159	0.3757	0.2037
200	0.4300	0.5131	1.5846	0.3214	0.6395	0.6473	0.3078	0.1993
300	0.4718	0.4849	1.5727	0.3182	0.6545	0.6046	0.2762	0.1934
Average					0.6711	0.6446	0.4576	0.2200

Table 4: Simulation results of set 2 with the LFZ spectral gradient conjugate method for 500 iterations

n	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$RMSE \hat{a}$	$RMSE \hat{b}$	$RMSE \hat{c}$	$RMSE \hat{\alpha}$
25	0.5567	1.1554	2.1864	0.2774	0.7168	1.8082	1.7081	0.4588
50	0.6068	1.0370	2.0044	0.2434	0.7768	1.1744	1.2371	0.2501
75	0.5418	1.1959	1.7892	0.2289	0.6711	1.1841	1.0262	0.1885
100	0.5329	1.1552	1.7008	0.2221	0.7009	1.1435	0.8013	0.1772
150	0.6705	1.1561	1.6481	0.2052	1.1649	1.0144	0.6200	0.1267
200	0.6040	1.2129	1.5552	0.2078	1.0015	1.6429	0.4657	0.1420
300	0.6279	1.2387	1.4974	0.1938	1.5366	1.4217	0.3772	0.1085
Average					0.9384	1.3452	0.8908	0.2074

**Figure 6:** RMSE per Parameter of the JYJLL (blue) and LFZ (red) Conjugate Spectral Gradient Methods for Set 2

Based on Table 1, Table 2, Table 3, Table 4, Figure 5 and Figure 6, the following is an analysis based on the RMSE value of the estimated parameters a , b , c , and α using the JYJLL and LFZ spectral gradient conjugate methods:

- For parameter a , the JYJLL method has a lower RMSE value compared to the LFZ method for sample sizes $n = 25, 50, 100, 150, 200$ and 300 . Based on Table 3 and Table 4, the average RMSE for all sample sizes n can be seen. The average RMSE for the JYJLL method is 0.6787 (set 1) and 0.6711 (set 2) and the average RMSE for the LFZ method is 0.7843 (set 1) and 0.9384 (set 2). Thus, it is obtained that the JYJLL method is better in estimating the parameter for the ExEW distribution compared to the LFZ method. Based on Figure 5 and Figure 6, it is obtained that the JYJLL method is more stable than the LFZ method.
- For parameter b , the JYJLL method has a lower RMSE value compared to the LFZ method for all sample sizes n . Based on Table 3 and Table 4, the average RMSE for all sample sizes n can be seen. The average RMSE for the JYJLL method is 0.6948 (set 1) and 0.6446 (set 2) and the average RMSE for the LFZ method is 1.8338 (set 1) and 1.3452 (set 2). Thus, it is obtained that the JYJLL method is better at estimating the b parameter for the ExEW distribution compared to the LFZ method. Based on Figure 5 and Figure 6, it is obtained that the LFZ method is unstable, meanwhile, the JYJLL method is more stable for all sample sizes n .
- For parameter c , the JYJLL method has a lower RMSE value compared to the LFZ method for all sample sizes n . Based on Table 3 and Table 4, the average RMSE for all sample sizes n can be seen. The average RMSE for the JYJLL method is 0.4321 (set 1) and 0.4576 (set 2) and the average RMSE for the LFZ method is 0.9028 (set 1) and 0.8908 (set 2). Thus, it is obtained that the JYJLL method is better at estimating parameter c for the ExEW distribution compared to the LFZ method. Based on Figure 5 and Figure 6, it is obtained that both methods experience a decrease along with the increasing number of samples n , with the RMSE value of the JYJLL method being smaller than the LFZ method. This shows that the estimation of parameter c is increasingly accurate with increasing sample size.
- For parameter α , the JYJLL method has a higher RMSE value compared to the LFZ method for sample sizes $n = 25$ and 100 (set 1) as well as for $n = 75, 100, 150, 200$ and 300 (set 2). Based on Table 3 and Table 4, the average RMSE for all sample sizes n can be seen. The average RMSE for the JYJLL method is 0.2269 (set 1) and 0.2200 (set 2) and the average RMSE for the LFZ method is 0.3212 (set 1) and 0.2074 (set 2). Thus, it is obtained that the JYJLL method is slightly worse in estimating the α parameter for the ExEW distribution compared to the LFZ method in set 2. However, based on Figure 5 and Figure 6, it is obtained that the JYJLL method remains more stable than the LFZ method.

Overall, the JYJLL spectral gradient conjugate method shows better and more consistent performance in estimating parameters a , b , and c . The LFZ method tends to have higher and unstable RMSE values for all parameters, except for parameter α . However, this indicates that the JYJLL method can be more reliable in estimating ExEW distribution parameters. It can be obtained that the results of both simulations show that the JYJLL spectral gradient conjugate method is better than the LFZ spectral gradient conjugate method in estimating the ExEW distribution parameters. In the next part, the JYJLL method is used to estimate the parameter values from real data.

4.2. Application in Airplane Windshield Failure Rate Data

Previously, a parameter estimation simulation of the ExEW distribution was carried out using the JYJLL method. Obtained from the analysis carried out, the JYJLL method provides better estimation results compared to the LFZ method. Therefore, the application of the ExEW model to real data uses the JYJLL method. In this section, the ExEW distribution is applied to simulation data that represents the airplane windshield failure rate. Failure rate is a measure used to describe the frequency of failure of an engineered system or component. Usually defined as the number of failures per unit time. Data obtained from the journal "The Extended Exponential Weibull Distribution: Properties, Inference, and Applications to Real-Life Data", 2022, which consists of 84 data.

Table 5: 84 Airplane Windshield Failure Rate Data (Thousands of Hours)

No	Data	No	Data	No	Data	No	Data	No	Data	No	Data	No	Data	No	Data
1	0.04	12	1.866	23	2.385	34	3.443	45	0.301	55	1.876	65	2.481	75	3.467
2	0.309	13	1.899	24	2.61	35	3.478	46	0.557	56	1.911	66	2.625	76	3.578
3	0.943	14	1.912	25	2.632	36	3.595	47	1.07	57	1.914	67	2.646	77	3.699
4	1.124	15	1.981	26	2.661	37	3.779	48	1.248	58	2.01	68	2.688	78	3.924
5	1.281	16	2.038	27	2.823	38	4.035	49	1.281	59	2.085	69	2.89	79	4.121
6	1.303	17	2.089	28	2.902	39	4.167	50	1.432	60	2.097	70	2.934	80	4.24
7	1.48	18	2.135	29	2.962	40	4.255	51	1.505	61	2.154	71	2.964	81	4.278
8	1.506	19	2.19	30	3.0	41	4.305	52	1.568	62	2.194	72	3.103	82	4.376
9	1.615	20	2.223	31	3.114	42	4.449	53	1.619	63	2.224	73	3.117	83	4.485
10	1.652	21	2.229	32	3.166	43	4.57	54	1.652	64	2.3	74	3.344	84	4.602

11	1.757	22	2.324	33	3.376	44	4.663
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The following provides descriptive statistics of the failure rate data in Table 5.

Table 6: Descriptive Statistics of Failure Rate Data

Descriptive Statistics	Value
Mean	2.55745
Median	2.3545
Modus	2.25
Variance	1.25177
Skewness	0.09949
Kurtosis	-0.65232
Min	0.04
Max	4.663

From Table 6, the difference between the median and mean values shows that the data distribution approaches a symmetrical shape. In addition, the skewness coefficient of 0.09949 indicates that the data tends to skew to the right. Kurtosis with a value of -0.65232 describes a flatter distribution curve shape compared to the normal distribution. Therefore, the data is suitable for modeling with the ExEW distribution. In addition, the data is also modeled with several other distributions, namely the Weibull distribution and the exponential distribution. The parameter estimation of these distributions is carried out using the MLE method with the help of the JYJLL spectral gradient conjugate method in RStudio software. The results of the ExEW, Weibull, and exponential distribution parameter estimates can be seen in Table 7 below.

Table 7: Parameter Estimator for Distributions Modeling Airplane Windshield Failure Rate Data

Model Distribution	Estimator	Estimation Results
Extended	$\hat{a}$	0.7911
exponential Weibull (ExEW)	$\hat{b}$	0.5580
	$\hat{c}$	2.9316
	$\hat{a}$	0.0645
Weibull	$\hat{a}$	0.0823
	$\hat{b}$	2.3744
Exponential	$\hat{a}$	0.3910

After obtaining parameter estimates from the ExEW, Weibull, and exponential distributions. Next, a model fit test is carried out on the data using the Kolmogorov-Smirnov test. In this case, the ExEW and Weibull distributions proven to be good in modeling airplane windshield failure rate data based on the Kolmogorov-Smirnov goodness-of-fit test. It has been found that the ExEW and Weibull distributions are suitable for modeling airplane windshield failure rate data. Therefore, it is necessary to select the best model. The first step taken in this study is to see the comparison of the PDF graphs of the ExEW and Weibull distribution models against the histogram of airplane windshield failure rate data. For the comparison of the PDF graph and data histogram, it can be seen in Figure 7 that the ExEW and Weibull distributions have almost the same graphs. It can also be seen from the shape of the data histogram, that the ExEW PDF graph is better at modeling airplane windshield failure rate data compared to the Weibull distribution. Then, seen from the comparison of the CDF graphs in Figure 7, the CDF that most closely matches the empirical CDF shape is the ExEW CDF. Therefore, with the graphical approach step, it can be obtained that the ExEW distribution is better at modeling airplane windshield failure rate data.

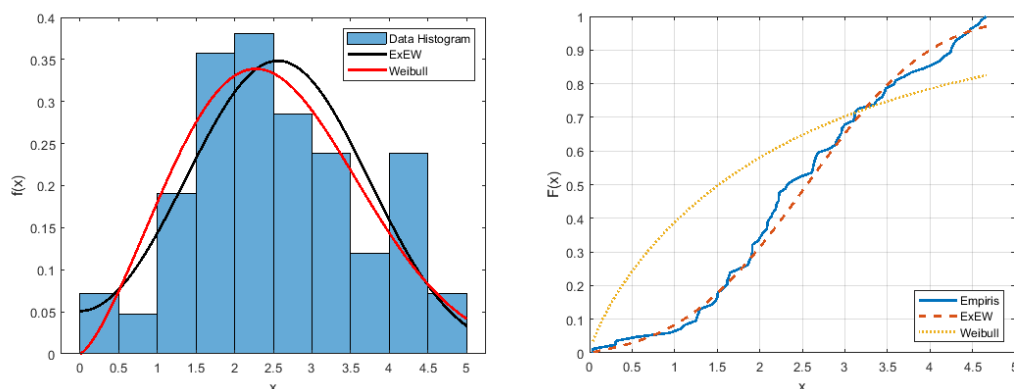


Figure 7: RMSE Comparison of PDF (left) and CDF (right) Plots of ExEW and Weibull Distributions with Histogram of Data

Next, the best model is selected by looking at the information criteria, namely AIC and BIC. The smaller the AIC and BIC values, the better the model is at modeling the data. Here are the AIC and BIC values for competing distributions:

Table 8: AIC and BIC Values for Each Distribution

Distribution Model	AIC	BIC
ExEW	261.3098	261.0069
Weibull	264.1066	263.9552

Based on Table 8 it is obtained that the ExEW distribution has the best AIC and BIC values, which are 261.3098 and 261.0069. Therefore, the ExEW distribution is better at modeling airplane windshield failure rate data compared to the Weibull distribution and the exponential distribution.

5. Conclusion

Based on the research conducted, the following conclusions can be drawn:

- The Jian-Yang-Jiang-Liu-Liu (JYJLL) spectral conjugate gradient method has proven to be effective in assisting the Maximum Likelihood Estimation (MLE) process for estimating the parameters of the Extended Exponential Weibull (ExEW) distribution. In data simulation studies, this method demonstrated strong performance, as indicated by low Root Mean Square Error (RMSE) values, even as the sample size increased.
- A comparative analysis of the JYJLL and Liu-Feng-Zou (LFZ) spectral conjugate gradient methods in simulation scenarios shows that the JYJLL method outperforms the LFZ method in parameter estimation. This is evidenced by the lower and more stable RMSE values obtained using the JYJLL method compared to the LFZ method.
- The distributional goodness-of-fit tests indicate that the ExEW distribution is the most suitable model for the given dataset compared to the competing distributions. This conclusion is supported by a Kolmogorov–Smirnov (K-S) test statistic below the critical value and the lowest AIC and BIC values among all candidate distributions.

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